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On shock impedance

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The objective of the paper is to formulate a physically realistic definition of shock impedance. The definition obtained, expression (15), is consistent with the acoustic limit, with the Polachek & Seeger result for head-on shock incidence, and with the various reflexion limits. It seems to be applicable to all known regular and irregular refractions, with the possible exception of a refraction containing a Guderley patch. It is shown that one result of increasing impedance mismatch is to produce precursor and postcursor waves. It is thought the impedance mismatch caused by the jets of supersonic aircraft might be used to reduce the sonic boom overpressure.

Introduction

The conventional definition of the acoustic impedance, z, of a gaseous medium, is the product of the density, ρ, and speed of sound, a, in the undisturbed medium:

$$z \equiv \rho a. \tag{1}$$

This quantity is of fundamental importance in the theory of sound refraction, and in fact refraction results whenever a propagating wave encounters a change in z. The classical problem in the theory of sound refraction considers two media of different z which meet along a plane boundary (interface). Then a plane wave i which starts in the first medium strikes the interface at some incidence angle μ_i. The angular direction of the wave is changed from μ_i to μ_t as it passes into the second medium. The relation between $\mu_{i,t}$ is given by Snell's law

$$\frac{a_i}{\sin\mu_i} = \frac{a_t}{\sin\mu_t}. \tag{2}$$

Evidently, (2) implies, *inter alia*, that the absolute velocities of i, t, along the interface have the same values,

$$v_i = v_t. \tag{3}$$

Now the boundary conditions require continuity in pressure, and particle displacement, normal to the interface. It can be shown that in order to satisfy these conditions, a reflected wave r must be propagated back into the first medium. The theory gives expressions for the transmission and reflexion coefficients which make it possible to compare the pressure changes across t, and r, with those across i. In this way it can be shown that r will be in phase, or out of phase, according as

$$\frac{\rho_i a_i}{\cos\mu_i} \lesseqgtr \frac{\rho_t a_t}{\cos\mu_t}. \tag{4}$$

At the equality condition the wave r is vanishingly weak. Expressions of the form appearing in this relation are sometimes called the 'effective impedance' (Morse & Ingard 1968), and they provide a more natural definition of $z_{i,t}$ for oblique refraction. Accordingly, the acoustic impedance will be defined, from here on, by

$$z_{i,t} \equiv \frac{\rho_{i,t}a_{i,t}}{\cos\mu_{i,t}}. \tag{5}$$

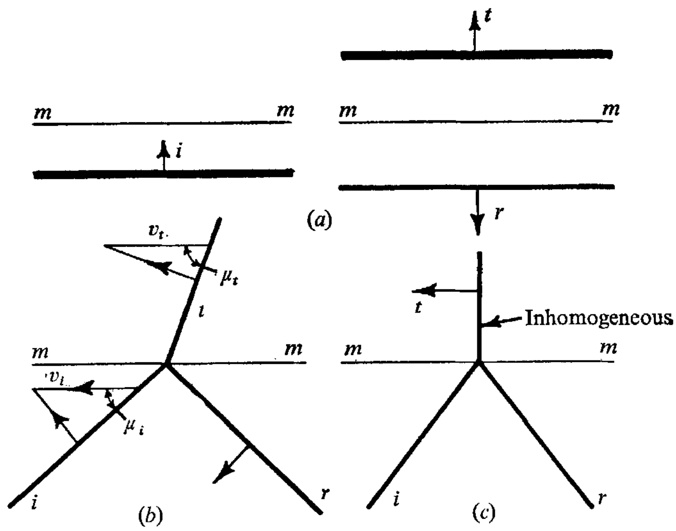

FIGURE 1. Effect of incidence angle μ_i on acoustic wave refraction: (*a*) $\mu_i = 0$, head-on collision; (*b*) $\mu_i > 0$, oblique refraction; (*c*) $\mu_t = \frac{1}{2}\pi$, total internal reflexion. *i*, incident wave; *r*, reflected wave; *t*, transmitted wave; *m–m*, gas interface.

An important parameter in the theory is μ_i and a systematic study of its effect can be made in the following way. Suppose initially that $\mu_i = 0$, which means that i makes a head-on collision with the interface, figure 1(*a*), then the waves t and r will also have zero wave angles $\mu_t = \mu_r = 0$. This type of refraction can either be regarded as being unsteady or as the limit of a steady system in which the waves propagate with infinite velocity parallel to the interface. The latter view is taken here. Next suppose that μ_i is continuously increased to some positive value, then the classical confluence of three waves will be obtained, figure 1(*b*). If the increase in μ_i is continued steadily, one eventually attains a condition where the theory gives an unreal solution. The amplitudes of i and r are then equal and one has the familiar total internal reflexion phenomenon. The wave t, although plane, is inhomogeneous, and it decays in amplitude as it retreats from the interface, figure 1(*c*). The real solutions correspond to wave systems containing only homogeneous waves, and these will be defined as *regular* refractions of i. The unreal solutions require at least one wave to be inhomogeneous, and will be defined as *irregular* refractions of i.

Of course it does not always follow that a change in the composition of the gas causes a change in z. But if special conditions exist in which z remains invariant with changing gas composition, then all the reflected waves must cancel mutually. For example, if a layer of gas of different composition is embedded in another gas, and if z is to be invariant, then following the double refraction, the two reflected waves must cancel to a wave of zero strength. Such transparencies are often of practical importance.

The above remarks have their counterpart in the refraction of shock waves, and in particular the concept of acoustic impedance has to be replaced by that of shock impedance. So far, a satisfactory definition of this quantity does not seem to be available, although Polachek & Seeger (1951) have successfully defined it for the limiting case of zero incidence $\omega_i = 0$. Their analogue of (4) can be written

$$a_i^{-1}\left[\gamma_i\left((\gamma_i+1)+(\gamma_i-1)\frac{P_0}{P_{1i}}\right)\right]^{\frac{1}{2}} \lesseqgtr a_t^{-1}\left[\gamma_t\left((\gamma_t+1)+(\gamma_t-1)\frac{P_0}{P_{1t}}\right)\right]^{\frac{1}{2}}, \qquad (6)$$

so that their definition of shock impedance, Z, is

$$\left.\begin{aligned} Z_{i,t} &\equiv a_{i,t}^{-1}\left[\gamma_{i,t}\left((\gamma_{i,t}+1)+(\gamma_{i,t}-1)\frac{P_{0,0}}{P_{1i,1t}}\right)\right]^{\frac{1}{2}}, \\ \omega_{i,t} &= 0. \end{aligned}\right\} \qquad (7)$$

At the acoustic limit where† amp $t \to$ amp $i \to 1$, expression (6) reduces to expression (4), as it should. The reflected wave will be a shock when $Z_i < Z_t$ and an expansion when $Z_i > Z_t$. The reflected wave degenerates to a Mach line at the equality condition.

The objective of the present paper is to formulate the appropriate definition of shock impedance when refraction takes place at an arbitrary angle of incidence. In considering what properties a physically satisfactory definition should have, one naturally requires that it should reduce to expression (5) at the acoustic limit and to expression (7) at zero wave incidence. Another limiting property which seems essential is that it should be consistent with reflexion phenomena. Thus, for example, if the rigidity of the second medium were to increase without limit, then this should correspond to the medium having infinite impedance, and to the reflected wave becoming a shock. On the other hand circumstances can exist where i can refract, without causing t to appear. A well-known example is in the refraction of a shock at the edge of a free jet which is diffusing into an initially still atmosphere. In this situation the impedance of the second medium should be unreal, and the reflected wave should become an expansion. These considerations suggest that it is the nature of the reflected wave which should determine if the impedance increases or decreases during refraction. Accordingly, we make the following fundamental assumption. *The shock impedance increases during refraction if the reflected wave is a shock, it decreases if the reflected wave is an expansion, but it remains unchanged if the reflected wave is a Mach line.* The initial conditions for the various wave phenomena can be determined with the help of the hodograph mapping technique (Henderson 1966) and the same technique will be used here to frame a consistent definition of shock impedance.

† The amplitude of a shock wave is defined to be its pressure ratio, for example, amp $i \equiv P_{1i}/P_0$.

Shock impedance for regular refraction

For regular refraction, as defined elsewhere (Polachek & Seeger 1951; Jahn 1956; Henderson 1966), all the waves in the system are locally straight and homogeneous in the neighbourhood of the refraction point. Analysis of the wave system makes use of three boundary conditions. First, if a stationary wave system is to exist, then all points on it must move with the same absolute velocity.

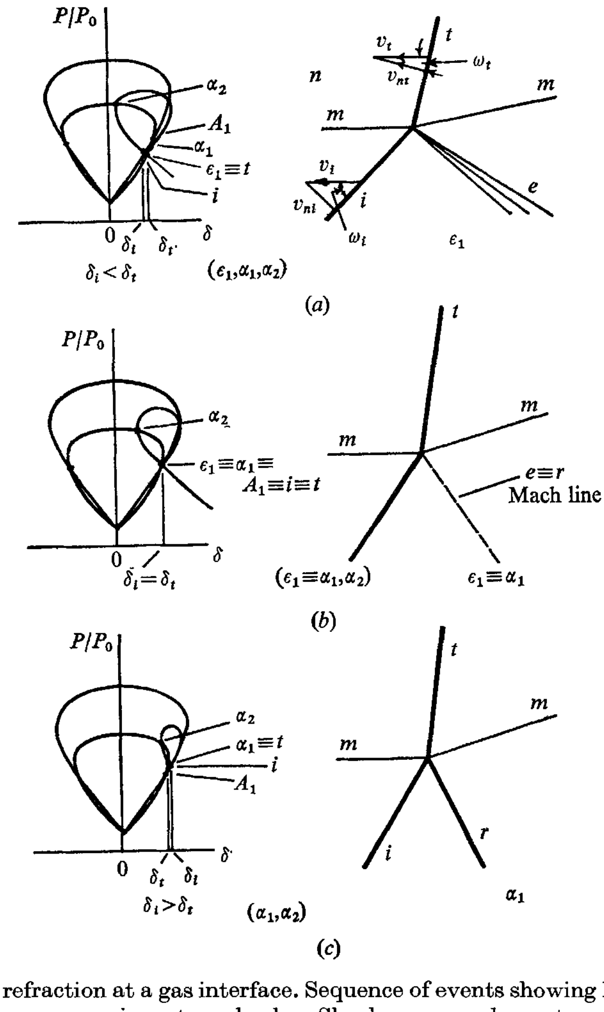

FIGURE 2. Regular refraction at a gas interface. Sequence of events showing how a reflected wave changes from an expansion e to a shock r. Shock wave angle ω, streamline deflexion angle δ, ordered set of solutions to three wave confluence (ϵ_1, α_1, α_2).

Hence in particular, along the undisturbed interface, one can write, figure 2(*a*),

$$v_i = v_t, \tag{8}$$

which implies that

$$\frac{v_{ni}}{\sin\omega_i} = \frac{v_{nt}}{\sin\omega_t}. \tag{9}$$

This of course is the analogue of Snell's law, but the law is now relaxed to the extent that the angle of reflexion is no longer equal to the angle of incidence. The other boundary conditions follow from the assumptions that the pressure and particle displacements are continuous across the deflected interface. These are conveniently written as

$$\frac{P_{1t}}{P_0} = \frac{P_{1i}}{P_0}\frac{P_{2i}}{P_{1i}} \tag{10}$$

and

$$\delta_t = \delta_i + \delta_r. \tag{11}$$

These boundary conditions, together with the shock wave equations and Prandtl–Meyer equation, provide a complete description of the regular refraction systems. The equations do, however, give more than one physically acceptable solution. But it has been found (Henderson 1966) that if all rigid boundaries are at infinity, then it is the weakest member of the ordered set of solutions that is physically relevant.

We now proceed to construct a sequence of events in regular refraction phenomena, during which the reflected wave changes from an expansion, to a shock, or *vice versa*. The example chosen for discussion occurs at a gas interface, and is of the slow–fast, $a_i < a_t$, variety.† This is quite a general problem and it includes a number of other problems as particular cases. In figure 2, the sequence is constructed for a wave i, incident at some angle ω_i. The maps change as the amplitude of i is continuously increased. From the hodograph diagrams it may be concluded that the reflected wave will be a shock, a Mach line, or an expansion, depending respectively on whether

$$\cot\delta_i \lesseqgtr \cot\delta_t, \tag{12}$$

where $\delta_{i,t}$ are the streamline deflexions across the incident and transmitted shocks. Hence by our basic assumption, this relation will determine if there is an increase, an equality, or a decrease in shock impedance. Now the equations of the shock hodographs (polars) are

$$\cot\delta_{i,t} = \frac{1+\gamma_{i,t}M^2_{0i,0t}-\dfrac{P_{1i,1t}}{P_{0,0}}}{\dfrac{P_{1i,1t}}{P_{0,0}}-1}\left[\frac{\dfrac{\gamma_{i,t}-1}{\gamma_{i,t}+1}+\dfrac{P_{1i,1t}}{P_{0,0}}}{\dfrac{2\gamma_{i,t}}{\gamma_{i,t}+1}M^2_{0i,0t}-\dfrac{\gamma_{i,t}-1}{\gamma_{i,t}+1}-\dfrac{P_{1i,1t}}{P_{0,0}}}\right]^{\frac{1}{2}}. \tag{13}$$

These equations can be substituted into (12), with the eventual result:‡

$$\left(1+\gamma_i M^2_{0i}-\frac{P_{1i}}{P_0}\right)\left[\frac{\dfrac{\gamma_i-1}{\gamma_i+1}+\dfrac{P_{1i}}{P_0}}{\dfrac{2\gamma_i}{\gamma_i+1}M^2_{0i}-\dfrac{\gamma_i-1}{\gamma_i+1}-\dfrac{P_{1i}}{P_0}}\right]^{\frac{1}{2}}$$
$$\lesseqgtr \left(1+\gamma_t M^2_{0t}-\frac{P_{1t}}{P_0}\right)\left[\frac{\dfrac{\gamma_t-1}{\gamma_t+1}+\dfrac{P_{1t}}{P_0}}{\dfrac{2\gamma_t}{\gamma_t+1}M^2_{0t}-\dfrac{\gamma_t-1}{\gamma_t+1}-\dfrac{P_{1t}}{P_0}}\right]^{\frac{1}{2}}. \tag{14}$$

† The fast–slow sequence is constructed with equal facility, and the results obtained are found to be consistent with what follows.

‡ Expression (14) has been simplified by using (10). Then the two inequalities, and the equality remain valid if the left-hand side of (14) is multiplied by $[(P_{1i}/P_0)-1]$, and the right-hand side by $[(P_{1t}/P_0)-1]$.

This last expression is more useful than (12), because it contains a property $\gamma_{i,t}$ of each gas, and also the amplitudes of i and t.

The quantities appearing in this expression will be defined as the shock impedance of the media, thus

$$Z_{i,t} \equiv \left(1+\gamma_{i,t} M^2_{0i,0t} - \frac{P_{1i,1t}}{P_{0,0}}\right)\left[\frac{\dfrac{\gamma_{i,t}-1}{\gamma_{i,t}+1}+\dfrac{P_{1i,1t}}{P_{0,0}}}{\dfrac{2\gamma_{i,t}}{\gamma_{i,t}+1}M^2_{0i,0t}-\dfrac{\gamma_{i,t}-1}{\gamma_{i,t}+1}-\dfrac{P_{1i,1t}}{P_{0,0}}}\right]^{\frac{1}{2}}. \tag{15}$$

The shock impedance of a gas is thus dependent on the amplitude of the wave in it. Now it will be recalled that the equations that describe a regular refraction give more than one physically acceptable solution. The ordered set of solutions will therefore require the second medium to have a different shock impedance for each member of the set. *In principle therefore it is possible to increase the impedance by placing suitable boundaries in the flow so as to force a stronger solution to appear.*

At the equality condition one has that $P_{1i}/P_0 = P_{1t}/P_0$, and expression (14) becomes a cubic in the wave amplitude. The algebraic complexity prevents us from giving the roots in a closed form. Some special cases have been worked out elsewhere; for example, the equality condition at a Mach number interface (Henderson & Macpherson 1968). The equality is the analogue of the transparency condition in acoustic refraction. The wave angle will in general change, even though the amplitude is constant.

The acoustic limit

At the acoustic limit, amp $i \to 1$, and by (15)

$$Z_{i,t} \to \frac{\gamma_{i,t} M^2_{0i,0t}}{(M^2_{0i,0t}-1)^{\frac{1}{2}}}, \tag{16}$$

or because $\sin \mu_{i,t} = M^{-1}_{0i,0t}$ and $a^2_{i,t} = \gamma_{i,t} P_{0,0}/\rho_{i,t}$, then

$$Z_{i,t} \to \frac{\gamma_{i,t} M_{0i,0t}}{\cos \mu_{i,t}} = \frac{\gamma_{i,t} v_{i,t}}{a_{i,t}\cos \mu_{i,t}} = z_{i,t}\frac{v_{i,t}}{P_{0,0}}. \tag{17}$$

On substituting this into (14) and making use of (8), one obtains $z_i \lesseqgtr z_t$, so that (14) and (15) are consistent with the acoustic limit.

The zero incidence limit, $\omega_i = 0$

This is the special case when i makes a head-on collision with the interface. As mentioned previously it can be considered to be the limiting condition where $M_{0i,0t} \to \infty$. Then with amp i, t finite, expression (15) becomes

$$\left.\begin{aligned} &Z_{i,t} \to \left(\tfrac{1}{2}v^2_{i,t}\frac{P_{1i,1t}}{P_{0,0}}\right)^{\frac{1}{2}} \times a^{-1}_{i,t}\left[\gamma_{i,t}\left((\gamma_{i,t}+1)+(\gamma_{i,t}-1)\frac{P_{0,0}}{P_{1i,1t}}\right)\right]^{\frac{1}{2}},\\ &\omega_{i,t} \to 0. \end{aligned}\right\} \tag{18}$$

On substituting these expressions into (14), and making use of (8) and (10), one immediately obtains the Polachek & Seeger result, expression (6).

The rigidity of the medium, B

Evidently, the refraction phenomena change into reflexion phenomena as the rigidity of the second medium becomes either 0, or ∞. We now make the further assumption that the bulk modulus B is the appropriate measure of the rigidity of a gas. This is defined in general terms by $B \equiv$ stress/strain, or for a gas

$$B \equiv -\frac{P_1 - P_0}{(V_1 - V_0)/V_0}. \tag{19}$$

Clearly this quantity depends on the nature of the process which changes the state of the gas, thus for an isentropic wave

$$B_{i,t} \equiv -V\frac{\partial P}{\partial V} = \rho_{i,t} a_{i,t}^2 \tag{20}$$

and this expression can be regarded as defining the acoustic bulk modulus. However, for a shock wave, the isentropic equation $PV^\gamma = K$, must be replaced by the Rankine–Hugoniot equation. This can be expressed in parametric form in terms of the Mach number component M_n normal to the shock, thus

$$\left.\begin{aligned} \frac{P_{1i,1t}}{P_{0,0}} &= \frac{2\gamma_{i,t}}{\gamma_{i,t}+1} M^2_{ni,nt} - \frac{\gamma_{i,t}-1}{\gamma_{i,t}+1}, \\ \frac{\rho_{1i,1t}}{\rho_{i,t}} &= \frac{(\gamma_{i,t}+1) M^2_{ni,nt}}{(\gamma_{i,t}-1) M^2_{ni,nt}+2}. \end{aligned}\right\} \tag{21}$$

On substituting these equations into (19), one obtains the following expressions for the shock bulk modulus

$$B_{i,t} \equiv \gamma_{i,t} P_{0,0} M^2_{ni,nt} = \rho_{i,t} v^2_{ni,nt}. \tag{22}$$

It is worth noting that for a particular gas $M_n^2 =$ (shock bulk modulus)/(acoustic bulk modulus). Hence the shock bulk modulus, just like the shock impedance, is dependent on the amplitude of the shock present in the gas.

The complaint limit, $Z_t \to B_t \to 0$, $M_{0t} \to 0$, and bound precursor waves

Suppose that $B_t \to 0$; this may happen in several ways, but for simplicity attention here is confined to the case where $M_{0t} \to 0$. The shock t degenerates to a sonic Mach line when $M_{0t} = 1$ and disappears altogether when $M_{0t} < 1$. It is easily confirmed from (15) that

$$\begin{aligned} &Z_t \to \infty, \quad \text{as} \quad M_{0t} \to 1, \\ &Z_t \text{ is unreal}, \quad \text{when } 0 < M_{0t} < 1, \\ &Z_t \to 0, \quad \text{as} \quad M_{0t} \to 0. \end{aligned}$$

The physical consequences of these changes need some comment. Thus consider the sequence of events illustrated in figure 3. The first pair of maps show a typical slow–fast irregular refraction, containing a Mach stem n. In this case there are two shocks refracted, namely j and k, and their respective refraction points are $F_{2,1}$.

In an earlier paper (Henderson 1966) it was shown that a necessary condition for the appearance of irregular refractions of the types illustrated in figure 3(a)

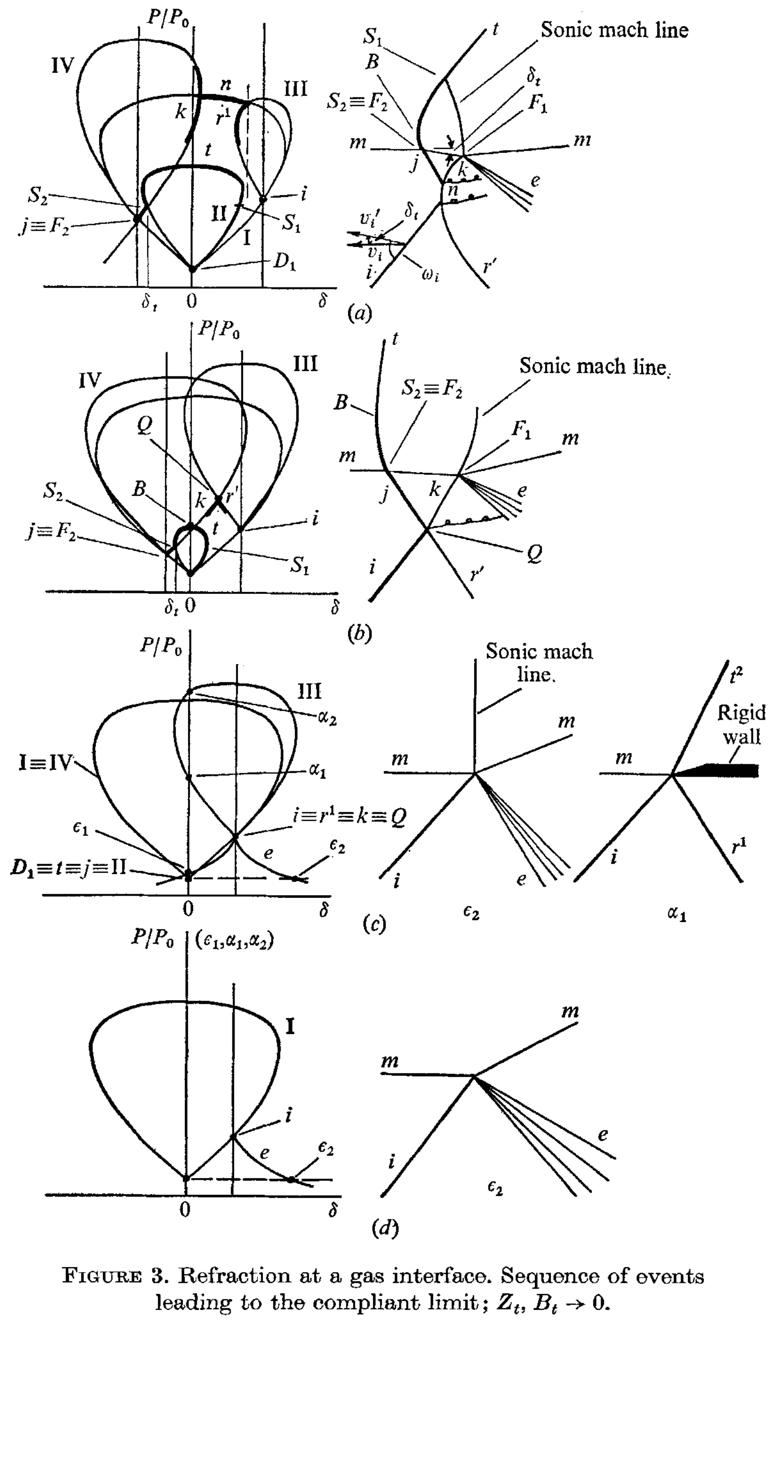

FIGURE 3. Refraction at a gas interface. Sequence of events leading to the compliant limit; Z_t, $B_t \to 0$.

(*b*), was that one or both of the boundary conditions (10), (11) should be violated or more specifically that $(P_{1i}/P_0)(P_{2i}/P_{1i}) > (P_{1t}/P_0)$, and $\delta_j+\delta_r > \delta_t$. As a result, t behaved as though it were a detached shock moving ahead of a blunt body. In the present context $B_t \to 0$, and $M_{0t} \to 1$, and therefore $Z_t \to \infty$. Then as transonic theory makes plain, t will move further and further ahead of F_1 until at $M_{0t} = 1$ the shock detachments, or stand-off, distance will be infinite. Now the waves j, r^1 of figure 3 (*b*) are due to t propagating from the second medium back into the first (feedback), so these waves must also move ahead with t. The totality of t, j, r^1, will be called the *precursor wave* of the refraction. If at some stage during this development Z_t, B_t are held constant for a time, then the precursor will of course take up a fixed position (the stand-off distance) ahead of F_1. Thus the precursor is stationary relative to i, k, e and for this reason it will be called a *stationary*, or *bound precursor*.

As $M_{0t} \to 1$, $\delta_t \to 0$, amp $j \to 1$, and the polar for t shrinks steadily. These changes cause both M_{0i} and M_{1j} to increase,† and eventually result in polars III and IV intersecting. It is this intersection which signals the change from the Mach reflexion system, figure 3(*a*), to the four-wave confluence system, figure 3(*b*). The next event of interest is the transitional condition, figure 3(*c*), where $M_{0t} = 1$, $P_{1t} = P_0$ and the precursor is at an infinite stand-off distance. The polar for t has shrunk into the point D_1 and all the waves in the precursor have become Mach lines. For the second medium the shock bulk modulus is now equal to the acoustic bulk modulus, while $Z_t = \infty$, and because of this $\delta_{F_1} = 0$. Another property of the condition is that the waves i, k have become part of the same shock which is simply relabelled i. The hodograph diagram indicates that there are three solutions that can satisfy the boundary condition $\delta_{F_1} = 0$, and these comprise the ordered set $(\epsilon_1, \alpha_1, \alpha_2)$. Now the ϵ_1 solution requires the expansion fan to be inclined forward of F_1, so this solution is rejected as being physically unrealistic. The $\alpha_{1,2}$ solutions are associated with a substantial pressure increase across the waves i, r^1, but this cannot be matched by the sonic Mach line whose amplitude is only unity. Hence the boundary condition cannot be satisfied for the $\alpha_{1,2}$ solutions either. There is, however, one rather artificial way in which the $\alpha_{1,2}$ solutions could be made physically acceptable. Thus suppose a rigid wall is placed along the interface and downstream of F_1, figure 3 (*c*), the wall is then able to support the higher downstream pressure associated with $\alpha_{1,2}$ and at the same time it ensures that $\delta_{F_1} = 0$. One would now expect to see the α_1 solution appear but it should also be possible to force α_2 to replace α_1, by placing another wall in the downsteam flow. This would need to be positioned so as to sustain the extra pressure associated with α_2. For example, it might be placed so as to create a second throat. However in the absence of these artificial aids, none of the solutions which satisfy $\delta_{F_1} = 0$ are physically acceptable. The $\delta_{F_1} = 0$ condition must therefore be invalid, and the constant pressure condition $P_{1t} = P_0$, imposes the requirement that the reflected wave is an expansion. One may now construct the ϵ_2 solution shown in figure 3(*c*). This solution remains valid as $M_{0t} \to 0$, and it is the familiar flow from an over-expanded jet, figure 3 (*d*).

† By equation (25), M_{0i} increases because $\delta_t \to 0$, and M_{1j} increases because amp $j \to 1$.

The singular behaviour at $M_{0t} = 1$ is plainly associated with the fact that B_t is finite while Z_t is infinite. It is this compliance of the second medium that enables one to either abandon the $\delta_{F_1} = 0$ condition, or else to retain it by making the medium artificially rigid with a suitably placed wall.

The rigid limit Z_t, $B_t \to \infty$

Suppose that one again starts with the maps shown in figure 3(*a*), (*b*) and it is desired to find out what happens as the speed of a wave of any amplitude in the second medium increases without limit, $a_t \to \infty$, $v_t \to \infty$. Then *both* B_t and Z_t will also increase without limit. If (8) is to remain valid during this process, then it will be necessary for the increase in v_t to be matched by an increase in v_i. The maps show that the wave system adjusts itself to meet the situation in the following way. First, it is noted that the streamlines which approach i are presented to the wave as though they are parallel† to F_1F_2, figure 4(*a*). Thus (8) can be extended to

$$v_i = v_i' \cos\delta_t = \frac{v_{ni}}{\sin\omega_i}\cos\delta_t = v_j = v_t, \tag{23}$$

where δ_t is measured at F_2, and ω_i is measured from F_1F_2. With increasing rigidity, a given stress will cause less and less strain in the second medium, hence $\delta_t \to 0$, and ω_i will become smaller. By (23), these changes will allow v_i to become larger, so that for a time, (8) can remain valid.

At F_1 the effect of the increasing rigidity is to cause the reflected wave to change from an expansion to a shock, because here also $\delta_{F_1} \to 0$. The result is an $\alpha_{1,2}$ solution in which the pressure change across the wave system in the first medium can now be supported by the increasing rigidity of the second medium. This leads to a shock reflexion of classical form, which may be either simple, or Mach, depending on the initial conditions, figure 4(*b*).

As the limit $\delta_t = 0$, and $v_t a_t = = \infty$, the Mach number M_{0t} will be restricted to being either 1 or ∞. The two values of M_{0t} are now considered in turn.

Free-precursor waves Z_t, $B_t \to \infty$, $M_{0t} \to 1$

An inevitable consequence of B_t, Z_t increasing without limit, while the velocity of i remains finite, is that sooner or later the adjustments indicated by (23) will be insufficient to maintain the validity of (8)‡ so that one will have:

$$\left.\begin{aligned} v_t &> v_i \\ \text{or}\qquad M_{0t}a_t &> M_{0i}a_i, \\ \text{or}\qquad a_t &> (M_{0i}/M_{0t}).a_i. \end{aligned}\right\} \tag{24}$$

For example, even the speed of an acoustic wave in the second medium a_t may become greater than the velocity of the incident shock along the interface.

† For simplicity it is assumed here that F_1F_2 is a straight segment. Experiments by Jahn (1956) seem to support this assumption. Detailed discussion of the structure of the interface in this region has been given by Henderson & Macpherson (1968).

‡ Only in the special case where $\omega_i = 0$ can (8) remain valid to the limit.

One may now infer the important result that when the inequality (24) is satisfied. the precursor is no longer a stationary wave but is free to propagate at a faster velocity along the interface, figure 4(*b*). The wave will then be called a non-stationary or *free* precursor. The velocity of j, r^1 along the interface are able to adjust to the increased velocity of t, by a reduction in wave angle, for example,

$$v_j = \frac{v_{nj}}{\sin \omega_j} = v_t \to a_t \to \infty. \tag{25}$$

It follows that M_{0j} will be increased because v_j has been increased while a_i has been held constant.

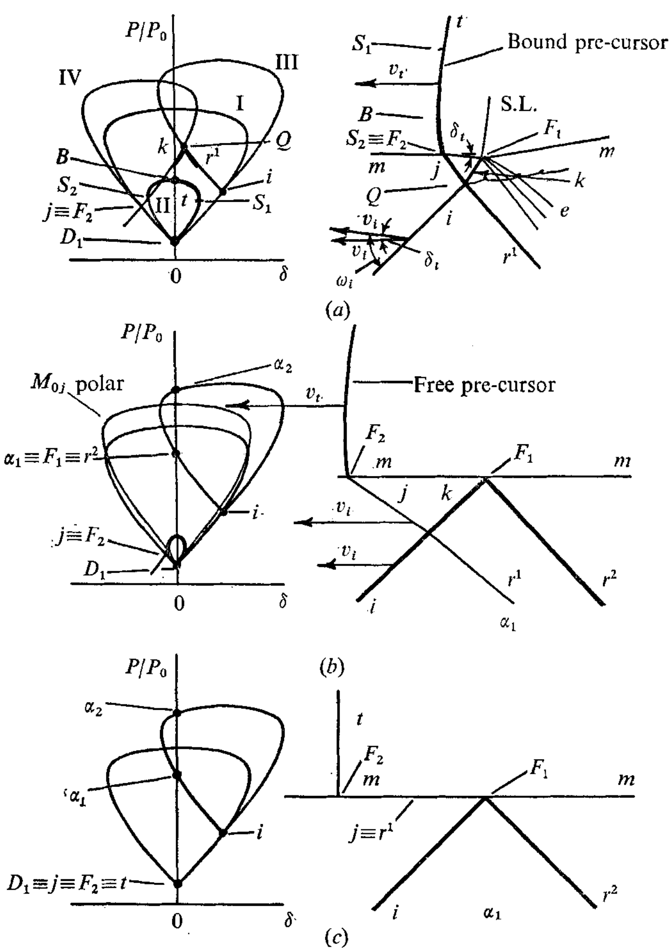

FIGURE 4. Irregular refraction at a gas interface. Sequence of events leading to $M_{0t} = 1$, type of rigid limit. (*a*) Bound precursor, $M_{0t} > 1$, $v_i = v_t$; (*b*) free precursor, $M_{0t} \to 1$, $v_i < v_t$; (*c*) free precursor, $M_{0t} = 1, v_i < v_t = \infty$.

From (24) it is evident that free precursors can be obtained for finite a_t, and therefore also for finite values of Z_t, B_t. This will mean in general that $M_{0t} > 1$ and amp $t > 1$ at the particular values of Z_t, B_t at which the precursors become free. For this situation, consider the initial shape of t, that is, soon after it has begun to move ahead of i. The hodograph diagram shows that t inclines forward of F_2 and then steepens to become a normal shock at B. At B one can write $v_t = v_{nt} \equiv v_{ntB}$ say. Beyond B the wave weakens and inclines backwards. Earlier papers (Henderson 1966 and Henderson & Macpherson 1968) indicate that the strong part of t, namely S_1BS_2, will be confined to a very small region at the interface, unless a suitable boundary† exists that would force S_1BS_2 to grow. Now with the passage of time the free precursors get further and further ahead of the stationary wave system until they have very little effect on it. Then at a given point in either gas there is a sudden increase in pressure as the precursors arrive. But the pressure falls as the waves move on, because the gases tend to return to their initial state. Expansion waves must therefore be present which decelerate the perturbed gases to zero velocity and to the undisturbed pressure. Additional expansion waves then accelerate the gases in the reverse direction as they tend to return to their original position. This means that a second precursor wave must exist, which once more restores the undisturbed pressure. One now has the familiar N wave, described by Whitham (1956).

The condition for the onset of the free precursors can now be put in a different form, thus by (23), (24) and (25)

$$v_i = \frac{v_{ni}\cos\delta_t}{\sin\omega_i} \leqslant v_t = v_{ntb}. \tag{26}$$

At the equality condition the precursors are bound to the stationary waves to form a typical refraction, for example, figure 4 (*a*), but at the inequality condition the precursors are set free and move ahead of the stationary waves. The refraction then consists of a stationary wave system, which in fact is a reflexion of either simple or Mach type, and a non-stationary system‡ which eventually develops into an N wave.

Finally, at the limiting condition $Z_t = B_t = v_t = a_t = \infty$, and $M_{0t} = 1$, the polar for t shrinks into the point D_1, figure 4 (*c*). Hence all precursors will then be Mach lines, with $\omega_t = \frac{1}{2}\pi$, and $\omega_i = \omega_{r^1} = 0$. The waves j, r^1 will therefore be coincident with the interface.

The so-called phenomenon of sonic cut-off is an example of precursors forming by refraction in gases. For consider a supersonic vehicle cruising at high altitude. Then under the action of increasing pressure and temperature, the impedance and the rigidity of the atmosphere increases from the aircraft to the ground. A condition can be reached, where the speed of sound near the ground is equal to the absolute speed of the aircraft. Below this relative sonic line, the boom signature is propagated upstream as precursors, which travel faster than the vehicle. The signature is also propagated in the downstream direction by post-cursor waves. Free precursors may also appear during the approach to the compliant limit. An example of this is to be found in the interaction of an oblique shock wave with a boundary layer. The Mach number inside the boundary layer declines from

† Such a boundary could be a local region in the second medium that has an even greater rigidity than the medium itself. It would also be necessary for it to have a suitable shape and position.

‡ The unsteady waves may be propagated in all directions in the second medium.

the free-stream value to zero at the wall, and this causes the shock to be refracted as it penetrates the boundary layer. At the wall $M_{0t} = B_t = Z_t = 0$, and one has the compliant limit. Any perturbation to the amplitude of the shock will produce both free- and post-cursor waves in the subsonic flow which exists below the boundary-layer sonic line.

Z_t, $B_t \to \infty$, and $M_{0t} \to \infty$

It was mentioned above that an irregular refraction such as the one shown in figure 4 (*a*), is formed by t moving ahead of F_1 much as though it were a detached shock taking up a position ahead of a blunt body. One way of causing this is to decrease M_{0t}, and in fact the stand-off distance increases very rapidly if $M_{0t} \to 1$ closely. Conversely, if M_{0t} begins to increase, then t will move in the downstream direction, so $F_2 \to F_1$, and the irregular system will soon revert to a regular system, figure 5 (*a*). Another consequence of increasing M_{0t} will be that its polar will shrink laterally and grow vertically,† until in the limit when $M_{0t} = \infty$, the polar will be coincident with the ordinate axis, and $\delta_t = 0$. With these changes Z_t and B_t again increase without limit, and once more a solution of the $\alpha_{1,2}$ type will appear.

One way in which this refraction can be envisaged is to consider the first medium to be at rest with the shock i propagating through it. At the same time, the second medium is considered to be moving along the interface at high velocity V, and coming from the direction of upstream infinity. Then while the second medium has some compliance, $B_t < \infty$, the waves i and r will cause it to be deflected at F_1, figure 5 (*b*) (much as though a wedge were introduced into the flow at F_1), and this will cause the shock t to appear. Consequently the equation‡ $v_i = v_t$ remains valid to the limit. At the limit $M_{0t} = B_t = \infty$, and for any i, r of finite amplitude $\delta_t = \omega_t = 0$, so that t will be coincident with the interface downstream of F_1, figure 5 (*c*).

Use of impedance mismatch to reduce the sonic boom overpressure

One way that this might be done is to make use of the impedance and the rigidity of the propelling jets. It would be necessary to place the engines underneath the aircraft in such a way that their jets intercept as much of the signature as possible. Then with the jet Mach number $M_j \to 1$, and with the jets made as hot as possible, say by the use of afterburners, the condition may be reached where the signature waves reflect off the top edges of the jets instead of refracting through them. Inside the jets, precursor and post-cursor waves will exist, and the jets act as a kind of wave guide for these unsteady waves. Of course these waves will themselves make disturbances at the bottom edges of the jets and this in turn will cause waves to be propagated towards the ground. If such a wave system were able to propagate far enough it would eventually organize itself into

† In the finite part of the plane.

‡ In the laboratory frame of reference. Here the velocity of t relative to the gas increases without limit because $v_{t\,\mathrm{rel.}} = v_{nt}/\sin\omega_t$, and $\omega_t \to 0$.

a somewhat unsteady N wave. The ultimate effect of the impedance mismatch is then to cause the aircraft to appear longer in the far field, so the overpressure is reduced by forcing the signature to be more spread out.

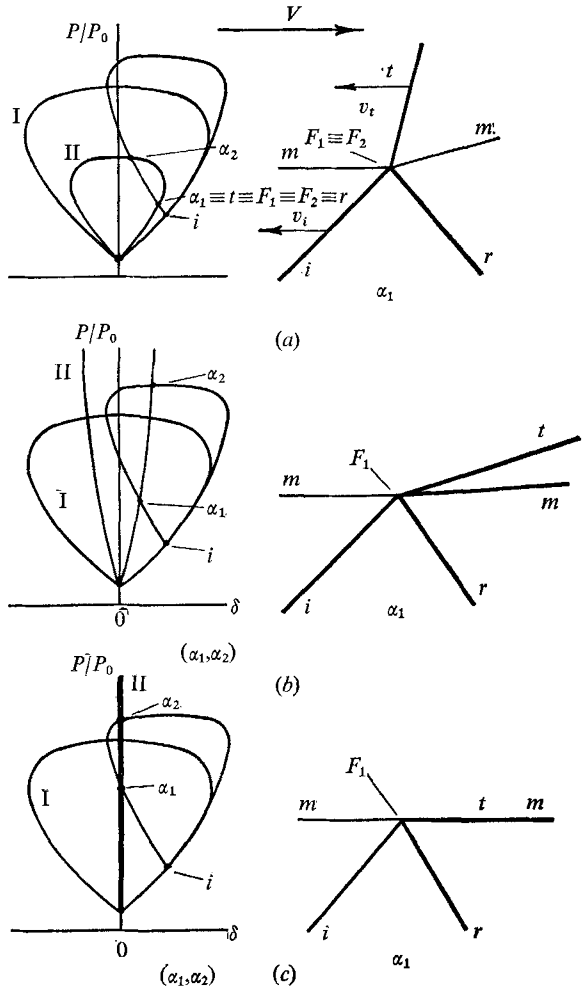

FIGURE 5. Regular refraction at a gas interface. Sequence of events leading to $M_{0t} = \infty$ type of rigid limit. (*a*) Regular refraction; (*b*) growth of polar for second medium; (*c*) rigid limit, $M_{0t} = B_{0t} = \infty$, $\omega_t = \delta_t = 0$.

Shock impedance for irregular refraction

The hodograph diagram indicates that there are basically four types of irregular refraction, figure 6. Two of these are of the slow–fast variety $a_i < a_t$, figure 6(*a*), (*b*), and the other two are of the reciprocal variety $a_i > a_t$, figure 6(*c*), (*d*). For the moment we note that for each variety the reflected wave is a shock for one system, figure 6(*a*), (*c*), and an expansion† for the other figure 6(*b*), (*d*). It is easy to show that the definition of shock impedance remains adequate when the reflected wave is an expansion; for referring to figure 6(*b*), (*d*), one has in

† The transitional condition is clearly ord $S_1 \equiv$ ord A_1.

both cases that $\cot\delta_i > \cot\delta_t$, and therefore by expressions (13), (14) and (15), $Z_i > Z_t$, as it should. Some difficulties are encountered, however, when the reflected wave is a shock. Thus for the refraction shown in figure 6(*a*), the wave *i* does not actually reach the interface; the waves *j* and *k* do so instead. If the impedance of the first medium is determined by *j*, then in most cases this gives $\cot\delta_i > \cot\delta_t$ and therefore $Z_j > Z_t$. This contravenes our basic assumption that the impedance should increase when the reflected wave is a shock. The position can be retrieved if one agrees to compare Z_i with $Z_{t\,\max}$, that is with the value of Z_t for which amp t is a maximum. Now amp t is a maximum at the point *B*, where *t* is normal to the flow. But the denominator of (15) is zero for a normal shock, hence† at *B*, $Z_t = Z_{t\,\max} = \infty$, and $Z_i < Z_{t\,\max}$. The streamline through *B* could be replaced by a straight wall, and when looked at in this way, it can be seen that it is the condition at *B* which is dominating the impedance of the second medium.

For the refraction shown in figure 6(*c*) there are two possibilities, either that $\cot\delta_i < \cot\delta_t$ in which case‡ $Z_i < Z_{t\,\max}$, which is consistent with our basic assumption, or that $\cot\delta_i > \cot\delta_t$ so that $Z_i > Z_{t\,\max}$, which is inconsistent. In the latter case the position can be retrieved to a large extent in the following way. It is noted that a Guderley patch is present in the flow (Guderley 1947). This means that both an expansion and a shock are reflected, and this circumstance is not covered by the basic assumption. The assumption is again applicable, however, once amp $i \geqslant$ ord S_1, for then at the equality condition the reflected waves become Mach lines and at the inequality condition they vanish altogether. As a result *i* and *n* become part of the same wave, and Z_i can be determined by the amplitude of the wave at A_1. This gives $\cot\delta_i = \cot\delta_t$ and $Z_i = Z_t$. The basic assumption is also applicable up to the point where the irregular refraction point forms. As shown elsewhere (Henderson & MacPherson 1968), it forms by $\alpha_{1,2}$ forming a double root, denoted by $(\alpha_1 \equiv \alpha_2)$, and then becoming unreal, figure 2(*c*). For the regular refraction, the reflected wave is a shock, and by figure 2(*c*), $\cot\delta_i < \cot\delta_t$, and so $Z_i < Z_t$ as it should. In summary, the only place where the basic assumption has been found to be applicable is for the small range of initial conditions ord $(\alpha_1 \equiv \alpha_2) <$ amp $i <$ ord S_1, which appears to be restricted to the fast–slow irregular refraction of the type illustrated in figure 6(*c*). This range can be considered a transitional situation, from a regular refraction where $Z_i < Z_t$ to an irregular refraction where $Z_i = Z_t$. Inside the range, one can write approximately $Z_i \simeq Z_t$.

Concluding remarks

The shock impedance *Z*, defined by expression (15), has been subjected to a fairly searching study, and it appears to be adequate for any circumstance likely to arise in practice. It has been shown to be consistent with the acoustic limit, with the Polachek & Seeger limit, and it is consistent with the various reflexion

† But the rigidity of the medium B_i remains finite.

‡ Here $Z_{t\max}$ occurs at the interface, and in general *t* nowhere becomes a normal shock.

limits. It serves all the known regular refractions, and all the irregular refractions, except perhaps for a small range of initial conditions where a Guderley patch is present. Even here though it should serve reasonably well. One can therefore assert with reasonable confidence, that a shock will refract whenever it encounters a change in Z, and further that the reflected wave will be a shock if Z increases and an expansion if it decreases.

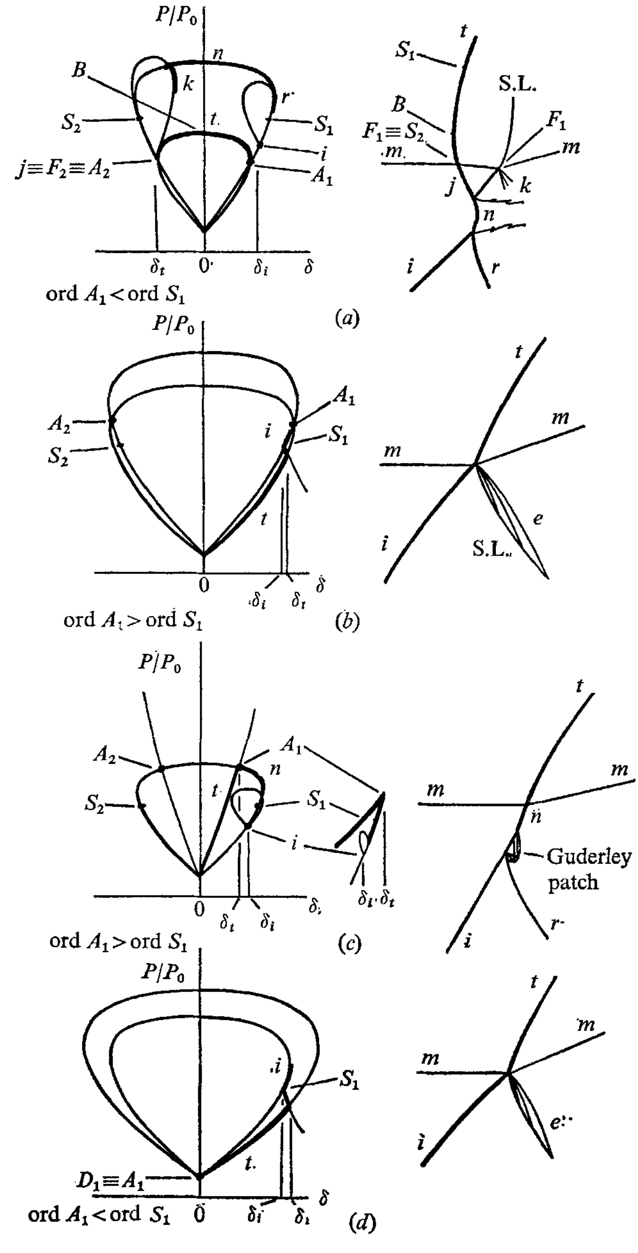

FIGURE 6. Basic irregular refractions at a gas interface, (*a*) slow–fast, $\delta_i > \delta_t$; (*b*) slow–fast, $\delta_i < \delta_t$; (*c*) fast–slow, $\delta_i > \delta_t$ or $\delta_t < \delta_i$; (*d*) fast–slow, $\delta_i < \delta_t$.

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